On the Enumeration of a Class of Non-Graceful Graphs

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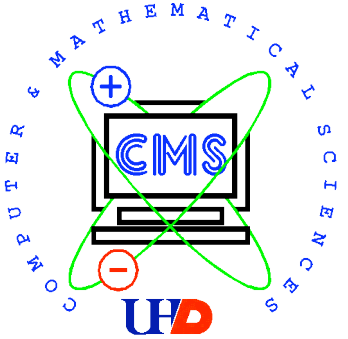


Graceful labelings of graphs

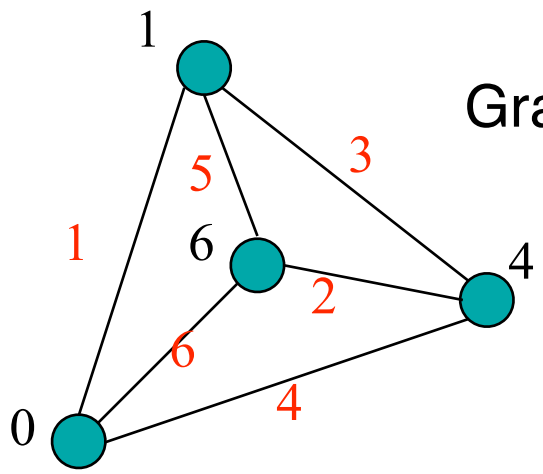
- A graceful labeling of $G = (V,E)$ is

$$f : V(G) \rightarrow \{0, 1, \ldots e\}$$

such that when each edge xy of $E(G)$ is assigned the label

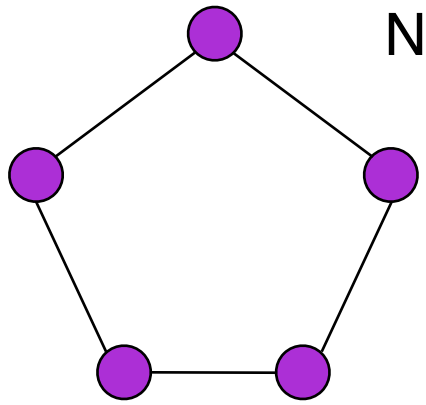
$$|f(x) - f(y)|$$

all edge labels are distinct



Graceful

- A graph G is a graceful graph if there exists a graceful labeling of G

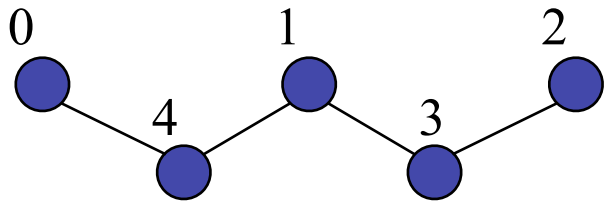


Not graceful

- Rosa (1967): “β-valuations”
- Golomb (1972): “graceful labeling”

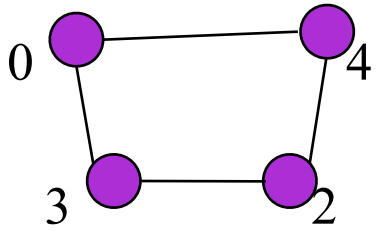
Some classes of graceful graphs

- All paths P_n are graceful (Rosa, 1967)



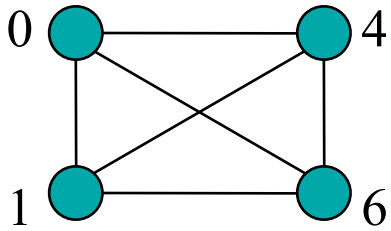
P_5

- Cycles C_n where $n \equiv 0$ or 3 (mod 4) are graceful (Rosa, 1967)



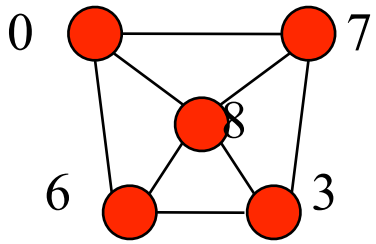
C_4

- Complete graphs K_n where $n \leq 4$ are graceful (Golomb, 1972)



K_4

- All wheels $W_n = C_{n-1} + K_1$ are graceful (Frucht, 1979)



$W_5 = C_4 + K_1$

<u>Ringel-Kotzig Conjecture</u>:
All trees are graceful (OPEN)

What about non-graceful graphs?

A sufficient condition for non-graceful graphs (Rosa, 1967):

> If *G* is a simple, even graph with *e* edges, and $e \equiv 1$ or 2 (mod 4), then *G* is <u>not</u> graceful.

We define **R** as the collection of all simple, even graphs with *e* edges, where $e \equiv 1$ or 2 (mod 4)

(**R** is an infinite proper subclass of the class of non-graceful graphs)

<u>Our goal</u>: Systematically enumerate graphs in **R** computationally

Our enumeration procedure

- Input: A value for e, where $e \equiv 1$ or $2 \pmod 4$
- Step 1: Generate all of the partitions of $2e$ consisting of only positive even integers, to produce a list of "number sequences"
- Step 2: Use the Havel-Hakimi procedure (1962) to determine which of these "number sequences" are graphic degree sequences
- Output: A list of graphic sequences that determine all of the graphs in **R** (both connected and not connected) with e edges

Examples when $e = 5$ and $e = 6$

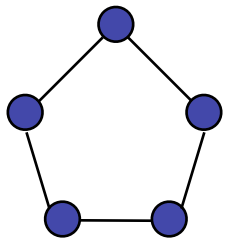
Input: $e = 5$

Results of Step 1 (Partitioning):

[10]
[8,2]
[6,4]
[6,2,2]
[4,4,2]
[4,2,2,2]
[2,2,2,2,2]

Results of Step 2 (Havel-Hakimi):

→ [2,2,2,2,2]



Input: $e = 6$

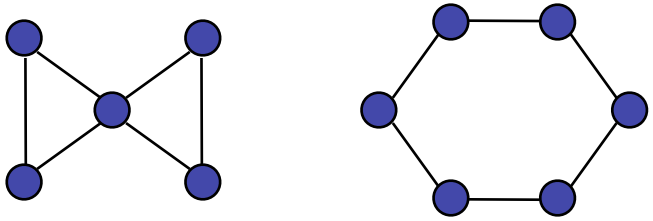
Results of Step 1 (Partitioning):

[12], [10,2],
[8,4], [8,2,2],
[6,6], [6,4,2],
[6,2,2,2], [4,4,4],
[4,4,2,2], [4,2,2,2,2],
[2,2,2,2,2,2]

Results of Step 2 (Havel-Hakimi):

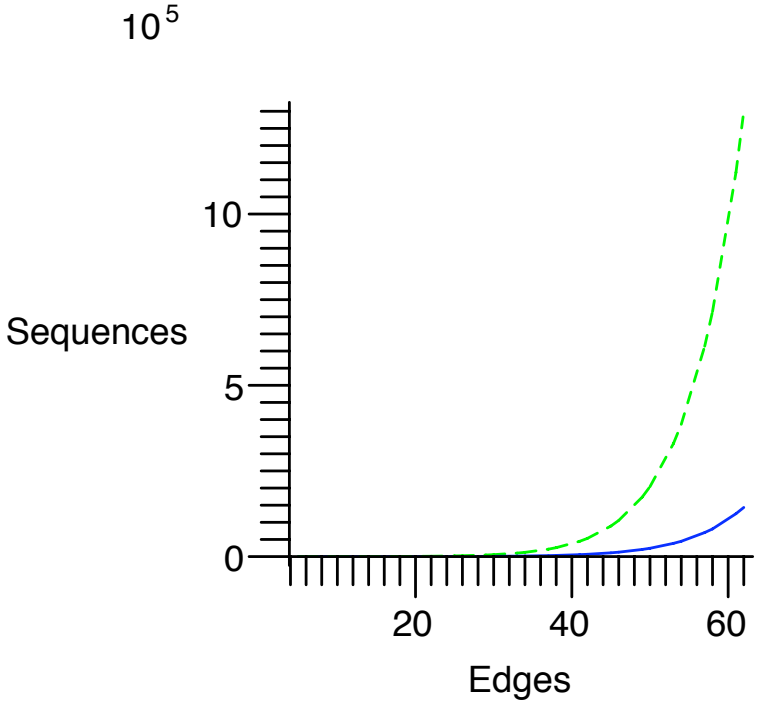
[4,2,2,2,2]
[2,2,2,2,2,2]

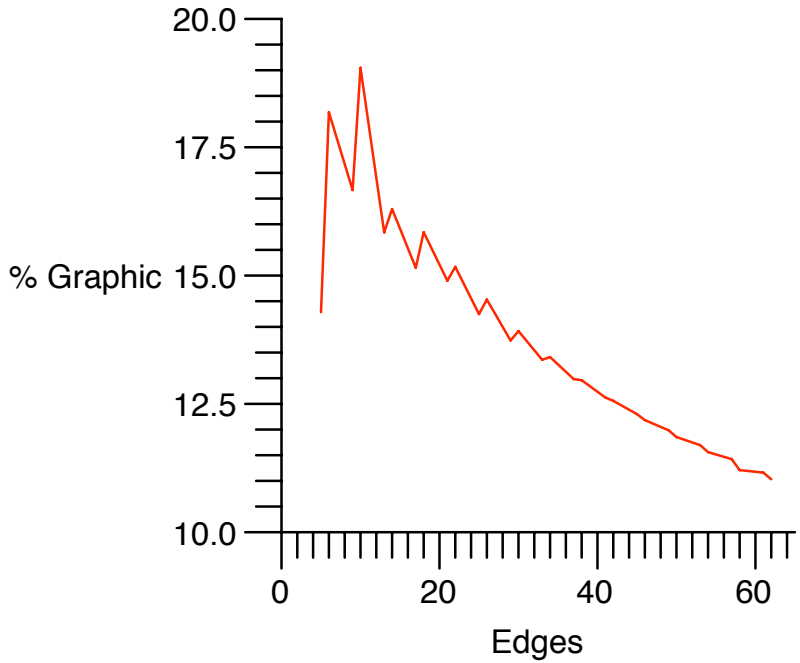
→



Computational Results

- What happens to the percentage of graphic sequences?





Total Sequences

Graphic Sequences

Conclusions and future work

- How high can we go? Working with larger and larger values of *e*
  - Improving our algorithms and/or implementation to obtain more results
  - Considering parallel computing to speed up procedure
  - Can always use faster and more powerful computers to obtain more results
- Initial observations/analysis from the table of results
  - Rapid growth in number of sequences and graphic sequences as *e* grows
  - Ever-so-slight decline in the percentage of graphic sequences as *e* grows
  - Computing time ("Total time" and "Partitioning time vs. Havel-Hakimi time")
- A related concept: "Disgracing a graceful graph"
  - Any graceful graph with *n* vertices and *e* edges can be "disgraced" (i.e., made non-graceful) by simply adding $e - n + 2$ isolated vertices to it.
  - Other ways to "disgrace" a graceful graph (e.g., if $n > 4$, "complete it")